UNIVERSITY OF CALIFORNIA PUBLICATIONS
IN
MATHEMATICS

Vol. 1, No. 17, pp. 371-387 November 8, 1923

A STUDY AND CLASSIFICATION OF RULED QUARTIC SURFACES BY MEANS OF A POINT-TO-LINE TRANSFORMATION

BY

BING CHIN WONG

UNIVERSITY OF CALIFORNIA PRESS
BERKELEY, CALIFORNIA

Note.—The University of California Publications are offered in exchange for the publications of learned societies and institutions, universities and libraries. Complete lists of all the publications of the University will be sent upon request. For sample copies, lists of publications or other information, address the Manager of the University of California Press, Berkeley, California, U. S. A. All matter sent in exchange should be addressed to The Exchange Department, University Library, Berkeley, California, U. S. A.

MATHEMATICS.—Derrick N. Lehmer, Editor. Price per volume, $5.00.

Cited as Univ. Calif. Publ. Math.

Vol. 1.
1. On Numbers which Contain no Factors of the Form $p(kp+1)$, by Henry W. Stager. Pp. 1–26. May, 1912 ... $0.50
2. Constructive Theory of the Unicursal Plane Quartic by Synthetic Methods, by Annie Dale Biddle. Pp. 27–54, 31 text-figures. September, 1912 ... .50
3. A Discussion by Synthetic Methods of Two Projective Pencils of Conics, by Baldwin Munger Woods. Pp. 55–85. February, 1913 ... .50
4. A Complete Set of Postulates for the Logic of Classes Expressed in Terms of the Operation "Exception," and a Proof of the Independence of a Set of Postulates due to Del Ré, by B. A. Bernstein. Pp. 87–96. May, 1914 ... .10
5. On a Tabulation of Reduced Binary Quadratic Forms of a Negative Determinant, by Harry N. Wright. Pp. 97–114. June, 1914 ... .20
6. The Abelian Equations of the Tenth Degree Irreducible in a Given Domain of Rationality, by Charles G. P. Kuschke. Pp. 115–162. June, 1914 ... .50
7. Abridged Tables of Hyperbolic Functions, by F. E. Pernot. Pp. 163–169. February, 1915 ... .10
8. A List of Oughtred's Mathematical Symbols, with Historical Notes, by Florian Cajori. Pp. 171–186. February, 1920 ... .25
9. On the History of Gunter's Scale and the Slide Rule during the Seventeenth Century, by Florian Cajori. Pp. 187–209. February, 1920 ... .35
10. On a Birational Transformation Connected with a Pencil of Cubics, by Arthur Robinson Williams. Pp. 211–222. February, 1920 ... .15
11. Classification of Involutory Cubic Space Transformations, by Frank Ray Morris. Pp. 223–240. February, 1920 ... .25
12. A Set of Five Postulates for Boolean Algebras in Terms of the Operation "Exception," by J. S. Taylor. Pp. 241–248. April, 1920 ... .15
13. Flow of Electricity in a Magnetic Field. Four Lectures, by Vito Volterra. Pp. 249–320, 40 figures in text. May, 1920 ... 1.25
14. The Homogeneous Vector Function and Determinants of the P-th Class, by John D. Barter. Pp. 321–343. November, 1920 ... .35
15. Involutory Quartic Transformations in Space of Four Dimensions, by Nina Alderton. Pp. 354–358. November, 1923 ... .25
16. On the Indeterminate Cubic Equation $x^3 + Dy^3 + D^2z^3 - 3\,Dxyz = 1$, by Clyde Wolf. Pp. 359–369. November, 1923 ... .25
17. A Study and Classification of Ruled Quartic Surfaces by means of a point-to-line Transformation, by Bing Chin Wong. Pp. 371–387. November, 1923 ... .25
18. A Special Quartic Curve, by Elsie Jeannette McFarland. Pp. 389–400, 1 figure in text. November, 1923 ... .25

PHILOSOPHY.—George P. Adams and J. Loewenberg, Editors. Volumes 1 and 2 are completed (price, $2.00 each); Volume 3 is in progress.

Vol. 1. Studies in Philosophy prepared in Commemoration of the seventieth birthday of Professor George Holmes Howison, November 29, 1904. 262 pages.

1. McGilvary, The Summum Bonum ... $0.25
2. Mezes, The Essentials of Human Faculty ... .25
3. Stratton, Some Scientific Apologies for Evil ... .15
4. Rieber, Pragmatism and the *a priori* ... .20
5. Bakewell, Latter-Day Flowing-Philosophy ... .20
6. Henderson, Some Problems in Evolution and Education ... .10
7. Burks, Philosophy and Science in the Study of Education ... .15
8. Lovejoy, The Dialectic of Bruno and Spinoza ... .35
9. Stuart, The Logic of Self-Realization ... .30
10. de Laguna, Utility and the Accepted Type ... .20
11. Dunlap, A Theory of the Syllogism ... .10
12. Overstreet, The Basal Principle of Truth-Evaluation ... .25

UNIVERSITY OF CALIFORNIA PUBLICATIONS
IN
MATHEMATICS

Vol. 1, No. 17, pp. 371-387 November 8, 1923

A STUDY AND CLASSIFICATION OF RULED QUARTIC SURFACES BY MEANS OF A POINT-TO-LINE TRANSFORMATION

BY
BING CHIN WONG

1. The purpose of this paper is to study and classify ruled quartic surfaces by means of a point-to-line transformation. The paper will consist of four sections:

Section I. A historical discussion of the subject.

Section II. The setting up of the machinery of transformation.

Section III. The study and classification of the ruled quartics of the species from I to IX.

Section IV. A slight modification of the machinery of transformation given in Section II and the classification of the remaining three species.

SECTION I

2. Ruled quartic surfaces have been studied and classified by Cremona,[1] Sturm,[2] Cayley,[3] Salmon,[4] and others.[5] These authors classify these surfaces according to their deficiency,[6] and reclassify them according to the nature of the double curves they have on them, and once more reclassify them according to the kinds of surfaces into which they are reciprocated.

3. There are two species of ruled quartics of deficiency unity and ten species of deficiency zero. The double curve on those of deficiency unity is a pair of straight lines which, if distinct give species I, and if coincident, give species II.[7] The

[1] Bologna Accad. Sci., *Mem.*, vol. 8 (1868)

[2] *Liniengeometrie,* Bd. 1, pp. 52-61

[3] *Phil. Trans.*, vols. 154 (1864) and 159 (1869).

[4] *Analytic Geom. of Three Dim.* (ed. 5), vol. 2, chap. 16.

[5] A discussion of ruled quartic surfaces may also be found in Jessop, *The Line Complex,* Cambridge, 1903, chap. 5 and in Basset, *The Geometry of Surfaces*, Cambridge, 1910, chap. 6.

[6] The deficiency of a surface is defined as the difference between the maximum and the actual number of double points on any plane section of the surface.

[7] This discussion is based upon the works of Cremona and Sturm, but the order of the species is Sturm's. Different authors number the classes differently. Owing to the difference in the point of view, the order in this paper is yet a different one.

double curve on those of zero deficiency is either a space cubic, or a straight line and a conic, or three straight lines. Those ruled quartics with a double space cubic are of species III if their reciprocal surfaces are of the same kind, and of species IV if their reciprocal surfaces have a triple line. Those whose double curve is a straight line and a conic belong to species V if they reciprocate into surfaces of the same kind; to species VI if they reciprocate into surfaces with a triple line. Those with three double straight lines make up the remaining six species: if the three lines one of which must be a generator are distinct, the surfaces belong to species VII; if two are coincident, the distinct one being the generator, the surfaces belong to species VIII. If the three double lines are all coincident, we have species IX if the surfaces reciprocate into those with a double place cubic; species X if they reciprocate into those of the same kind; species XI if the triple line is itself a generator counted once; and finaly, species XII if the triple line is itself a generator counted twice. Species XI reciprocate into surfaces with a straight line and a conic, and species XII into those of the same kind.[8]

4. The table[9] given (p. 373) presents a view of the different orders given to the species by Sturm, Cremona, Cayley, and Salmon, and also exhibits the special features of the different classes. In this table

k^3 stands for double space cubic;
c^2 stands for double conic;
e stands for double generator;
d stands for double directrix;
$d+d$ stands for two coincident double directrices;
$d+d'$ stands for two distinct double directrices;
$d+d+d$ stands for a triple line,
$d+d+d_e$ stands for a triple line which is itself a generator counted once;
$d+d_e+d_e$ stands for a triple line which is itself a generator counted twice.

The double curves on the reciprocal surfaces are denoted by the corresponding Greek letters.

It is obvious that species IV and VI reciprocate into species IX and XI respectively, and vice versa; while the other species are their own reciprocals.

5. The classification given in the following paragraphs is from a different point of view. Here we start with certain plane curves[10] which, by a point-to-line transformation, go into quartic surfaces, and, it will be shown, all the different classes of ruled quartics can be obtained in this manner. It will also be shown that there exists a close relation between the singularities[11] on the surfaces and the position and class of the plane curves.

[8] Salmon divides those surfaces with a triple line into five classes, I—V according to his enumeration, making thirteen species in all, while Sturm divides them into four, IX—XII as given above. As a matter of fact, V of Salmon is only a subform of IV, that is, a subform of XII of Sturm. See § 47 below.

[9] This table, with a slight modification and an addition of Salmon's enumeration, is the combination of the two found in Jessop, *The Line Geometry*, p. 80.

[10] For our purposes, conics and plane cubics only need be employed. Plane quartics may be used but they offer nothing new. No quintic curve or curve of higher degree can give quartic surfaces. But if one were to study and classify ruled surfaces of higher degree from this point of view, one would have to resort to curves of higher order, both plane and space.

[11] Including double curves and pinch-points.

6. This method has three interesting features: (1) the one-to-one correspondence between the generators of the surfaces and the points on the plane curves; (2) the dependence of the singularities on the surfaces upon the position and class of the plane curves; and (3) the possibility of the method in studying and classifying scrolls of higher order.

Def.	Double curve	Double curve on rec. s.	Sturm	Cremona	Cayley[12]	Salmon[13]
$p=1$	$d+d'$	$\delta+\delta'$	I	11	1	11
	$d+d$	$\delta+\delta$	II	12	4	13
$p=0$	k^3	k^3	III	1	10	6
	k^3	$\delta+\delta+\delta$	IV	7	8	7
	c^2+d	$k^2+\delta$	V	2	7	8
	c^2+d	$\delta+\delta+\delta_\epsilon$	VI	4	(11)	9
	$d+d'+e$	$\delta+\delta'+\epsilon$	VII	5	2	10
	$d+d+e$	$\delta+\delta+\epsilon$	VIII	6	5	12
	$d+d+d$	k^3	IX	8	9	1
	$d+d+d$	$\delta+\delta+\delta$	X	9	3	2
	$d+d+d_e$	$k^2+\delta$	XI	3	(12)	3
	$d+d_e+d_e$	$\delta+\delta_\epsilon+\delta_\epsilon$	XII	10	6	4 (5)

SECTION II

7. Take two fixed quadrics

$$F_1\equiv\sum_1^4 a_i x^2_i=0,$$

$$F_2\equiv\sum_1^4 b_i x^2_i=0.$$

Any point $Y(y_1, y_2, y_3, y_4)$ in space is transformed into a line

$$l\equiv\begin{cases}\sum_1^4 a_i y_i x_i=0,\\ \sum_1^4 b_i y_i x_i=0,\end{cases}$$

[12] Cayley does not consider (11) and (12) as separate species, but as subforms of 8 and 9 respectively.

[13] Salmon begins with surfaces having the highest singularity, that is, a triple line, and divides them into five classes, and then takes up those having a proper space cubic, and then those whose cubic degenerates, etc. See note 8.

which is the intersection of the polar planes of Y with respect to F_1 and F_2. To the ∞^3 of points in space there ∞^3 of polar planes with respect to F_1 and also ∞^3 of polar planes with respect to F_2 corresponding. The space consisting of the ∞^3 of polar planes with respect to F_1 and the space consisting of the ∞^3 of polar planes with respect to F_2 are clearly collineated to each other, for to every plane of the one space there corresponds a plane of the other space, both being polar planes of the same point. Then every line 1, being a line of the intersection of the corresponding planes of the two collineated spaces, is a line of a tetrahedral complex[14] whose fundamental tetrahedron $A_1A_2A_3A_4$ is the self-polar tetrahedron common to F_1F_2. To show this algebraically [15], write the six homogeneous coördinates of l:

$$q_{12} : q_{13} : q_{14} : q_{34} : q_{42} : q_{23} = \pi_{12}y_1y_2 : \pi_{13}y_1y_3 : \pi_{14}y_1y_4 : \pi_{34}y_3y_4 : \pi_{42}y_4y_2 : \pi_{23}y_2y_3$$

where $\pi_{ik}=a_ib_k-a_kb_i$. Then

$$\frac{q_{42}q_{13}}{q_{14}q_{23}}=\frac{\pi_{42}\pi_{13}}{\pi_{14}\pi_{23}},$$

$$\frac{q_{14}q_{23}}{q_{12}q_{34}}=\frac{\pi_{14}\pi_{23}}{\pi_{12}\pi_{34}},$$

and

$$\frac{q_{12}q_{34}}{q_{13}q_{42}}=\frac{\pi_{12}\pi_{34}}{\pi_{13}\pi_{42}},$$

Clearing fractions and adding, we have

$$Aq_{12}q_{34}+Bq_{13}q_{42}+Cq_{14}q_{23}=0,$$

where

$$A=\pi_{13}\pi_{42}-\pi_{14}\pi_{23},$$
$$B=\pi_{14}\pi_{23}-\pi_{12}\pi_{34},$$
$$C=\pi_{12}\pi_{34}-\pi_{13}\pi_{42},$$

which is the equation of the tetrahedral complex. Since the coördinates of the point Y are not involved in this equation, *every point in space goes into a line belonging to the above complex determined by the fixed quadrics.*

8. If a point describes a straight line in space, its two polar planes with respect to F_1 and F_2 describe two projective axial pencils whose corresponding planes meet in the generators on a ruled quadric. Algebraically, let

$$(y_1+\lambda z_1,\ y_2+\lambda z_2,\ y_3+\lambda z_3,\ y_4+\lambda z_4)$$

be the coördinates of any point on a given line g through Y and Z. Then the intersections of the corresponding planes of the two projective axial pencils determined by the given line g are given by

$$\sum_1^4 a_i(y_i+\lambda z_i)x_i=0,$$

$$\sum_1^4 b_i(y_i+\lambda z_i)x_i=0.$$

[14] Reye, *Geometrie der Lage*, Abth. 2, erste Auflage (1868), 15. Vortrag. An exposition of this complex is also given by Sturm in his *Liniengeometrie*, Bd. 1, pp. 332-382.

[15] Jessop, *The line Complex*, Cambridge, 1903, chap. 7.

Eliminating λ, we have

$$\sum_{i=1\, k=1}^{4} \pi_{ik}p_{ik}x_ix_k \equiv \pi_{12}p_{12}x_1x_2+\pi_{13}p_{13}z_1x_3+\pi_{14}p_{14}x_1x_4+\pi_{34}p_{34}x_3x_4+\pi_{42}p_{42}x_4x_2+\pi_{23}p_{23}x_2x_3 =0,$$

[where $p_{ik}=y_iz_k-y_kz_i$] which is the equation of a quadric surface. But the p_{ik} are the six homogeneous coördinates of the given line, and therefore, we conclude that *when the p_{ik} of any line are given we can at once write down the equation of its corresponding quadric in the above form.*

9. Now if we take three straight lines p'_{ik}, p''_{ik}, p'''_{ik} forming a triangle in space whose corresponding quadrics are, respectively,

$$Q' \equiv \sum_1^4 \pi_{ik}p'_{ik}x_ix_k=0,$$

$$Q'' \equiv \sum_1^4 \pi_{ik}p''_{ik}x_ix_k=0,$$

$$Q''' \equiv \sum_1^4 \pi_{ik}p'''_{ik}x_ix_k=0,$$

any conic

$$A_{11}p'^2_{ik}+A_{22}p''^2_{ik}+A_{33}p'''^2_{ik}+2A_1A_2p'_{ik}p''_{ik}+2A_2A_3p''_{ik}p'''_{ik}+2A_3A_1p'''_{ik}p'_{ik}=0$$

in the plane of this triangle will go into a ruled quartic surface whose equation is

$$A_{11}Q'^2+A_{22}Q''^2+A_{33}Q'''^2+2A_{12}Q'Q''+2A_{13}Q'Q'''+2A_{23}Q''Q'''=0$$

or

$$\sum_1^4 \pi_{ik}\pi_{jl}[\Sigma A_{rs}p^{(r)}{}_{ik}p^{(s)}{}_{jl}]x_ix_kx_jx_l=0 \qquad [r,\ s=1,\ 2;\ A_{rs}=A_{sr}].$$

A plane cubic in the plane of the same triangle will go into a sextic surface whose equation is

$$\Sigma A_{rst}Q^{(r)}Q^{(s)}Q^{(t)} \equiv$$

$$\Sigma \pi_{ik}\pi_{jl}\pi_{mn}[\Sigma A_{rst}p^{(r)}{}_{ik}p^{(s)}{}_{jl}p^{(t)}{}_{mn}]x_ix_kx_jx_lx_mx_n=0$$

$$[A_{rst}=A_{rts}=A_{srt}=A_{str}=A_{tsr}=A_{trs};\ r,\ s,\ t=1,\ 2,\ 3].$$

Similarly, a plane curve of degree n goes into a ruled surface of degree $2n$. All the surfaces thus obtained have the four vertices of the fundamental tetrahedron in common.

10. Corresponding to the ∞^2 of points in a plane u is a congruence of lines all cutting across a cubic k^3 twice. To prove this, when u is given, we have two points Z and Z' fixed which are the poles of u with respect to F_1 and F_2. Z and Z' can be considered as two projective point systems, and the locus of the intersections of the corresponding rays which do meet is a space cubic. Every point in u will have its two polar planes, one through Z and the other through Z', intersecting in a line cutting across the cubic twice. Since every line in u has its corresponding quadric

through the four vertices A_1, A_2, A_3, A_4 of the tetrahedron of reference, this cubic k_3 must go through them also, and is therefore a cubic of the complex. Thus, *every plane in space determines a cubic of the complex uniquely. Every plane curve goes into a surface on which lies the cubic determined by the plane of the curve.*

11. A pair of intersecting lines is transformed into a pair of quadrics having in common a generator which comes from the point of intersection of the two given lines and cubic k^3 which is determined by the plane of the two given lines. To obtain the parametric equations of k^3, let (u_1, u_2, u_3, u_4) be the coördinates of a given plane u, and let p'_{ik} and p''_{ik} be any two lines lying in it. The two quadrics corresponding to these two lines

$$Q' \equiv \sum_1^4 \pi_{ik} p'_{ik} x_i x_k = 0,$$

$$Q'' \equiv \sum_1^4 \pi_{ik} p''_{ik} x_i x_k = 0,$$

have, besides the common ruling into which the intersection of p'_{ik} of p''_{ik} goes, a common cubic k^3 whose parametric equations are easily found to be, with λ as the parameter,

$$\begin{aligned}
\rho x_1 &= u_1(a_2\lambda - b_2)\ (a_3\lambda - b_3)\ (a_4\lambda - b_4),\\
\rho x_2 &= u_2(a_3\lambda - b_3)\ (a_1\lambda - b_1)\ (a_1\lambda - b_1),\\
\rho x_3 &= u_3(a_4\lambda - b_4)\ (a_1\lambda - b_1)\ (a_2\lambda - b_2),\\
\rho x_4 &= u_4(a_1\lambda - b_1)\ (a_2\lambda - b_2)\ (a_3\lambda - b_3),
\end{aligned}$$

or

$$\rho x_i = u_i \overset{4}{\underset{1}{P}}(a_j\lambda - b_j) \qquad [i=1, 2, 3, 4;\ i \neq j].$$

These equations are independent of the coördinates of the lines but not of those of the given plane; therefore, when the coördinates of any plane are given, the equations of its corresponding cubic can be easily written.

12. Now if $u_1 = 0$, i.e., if the plane u passes through the vertex A_1 of the fundamental tetrahedron, k^3 degenerates into a conic c^2 lying in the plane x_1 and a straight line d which passes through A_1 and cuts across the conic. c^2 and d must have a point in common, for, if not, the quadric which corresponds to a line in u $(0, u_2, u_3, u_4)$ and therefore contains c^2 and d, would be intersected by a straight line lying in x_1 through P in which d pierces x_1 in three points, one being P itself and the other two being on c_2. This is impossible.

13. We can get the parametric equations of the conic c_2 by putting $u_1 = 0$ in the parametric equations of k^3 (Art. 11), or we can get them as follows:

Let (o, u_2, u_3, u_4) be the coördinates of the plane u and let any other plane $u'(u'_1, u'_2, u'_3, u'_4)$ intersect it in the line

$$q_{ik}(u'_1 u_2 : u'_2 u_3 : u'_4 : q_{34} : q_{43} : q_{23}).$$

Then corresponding to this line we have the quadric

$$\pi_{12}q_{34}x_1x_2+\pi_{13}q_{42}x_1x_3+\pi_{14}q_{23}x_1x_4 \\ -u'(\pi_{23}u_4x_2x_3+\pi_{42}u_3x_2x_4+\pi_{34}u_2x_3x_4)=0.$$

Putting $x_1=0$, we have the conic in which this quadric is intersected by the plane x_1:

$$x_1=0,\ \pi_{23}u_4x_2x_3+\pi_{42}u_3x_2x_4+\pi_{34}u_2x_3x_4=0.$$

These equations are independent of the coördinates of the plane u', and therefore represent the conic common to all the quadrics corresponding to all the lines in the plane u. This conic is none other than c^2, part of the degenerate cubic k_3.

14. Now to obtain d: Take two lines through A_1:

$$l_1\equiv\begin{cases} u(o,\ u_2,\ u_3,\ u_4), \\ u'(o,\ u'_2,\ u'_3,\ u'_4), \end{cases}$$

$$l_2\equiv\begin{cases} u(o,\ u_2,\ u_3,\ u_4), \\ u''(o,\ u''_2,\ u''_3,\ u''_4), \end{cases}$$

whose corresponding quadrics (pairs of planes)

$$Q_1\equiv x_1[\pi_{12}q'_{34}x_2+\pi_{13}q_{42}x_3+\pi_{14}q'_{23}x_4]=0,$$
$$Q_2\equiv x_2[\pi_{12}q''_{34}x_2+\pi_{13}q''_{42}x_3+\pi_{14}q''_{23}x_4]=0,$$

have, besides the plane x_1, the line

$$\pi_{12}q'_{34}x_2+\pi_{13}q'_{42}x_3+\pi_{14}q'_{23}x_4=0,$$
$$\pi_{12}q''_{34}x_2+\pi_{13}q''_{42}x_3+\pi_{14}q''_{23}x_4=0,$$

in common. This line has its homogeneous coördinates:

$$0:0:0:\pi_{13}\pi_{14}u_2:\pi_{12}\pi_{14}u_3:\pi_{12}\pi_{13}u_4$$

and has its parametric representations:

$$\rho x_1=\lambda\pi_{12}\pi_{14}u_3,$$
$$\rho x_2=\ \pi_{13}\pi_{14}u_2,$$
$$\rho x_3=\ \pi_{12}\pi_{14}u_3,$$
$$\rho x_4=\pi_{12}\pi_{13}u_4.$$

Both the coördinates and the representations of this line are independent of the coördinates of the plane u' and those of the plane u'', and therefore this line is the line d common to all the quadrics coming from all the lines in the plane u.

15. Now if $u_1=u_2=0$, i.e., if the plane u passes through the edge A_1A_2 of the fundamental tetrahedron, we have its k^3 broken up into three straight lines $d+d'+e$. d lies in the plane x_1, d' in x_2, while e being the intersection of x_1 and x_2 cuts across the other two. Let any plane

$$u'(u'_1,\ u'_2,\ u'_3,\ u'_4)$$

intersect the plane u $(0,\ 0,\ u_3,\ u_4)$ in the line whose transform is the quadric

$$\pi_{12}q_{34}x_1x_2+\pi_{13}u_4u'_2x_1x_3-\pi_{14}u_3u'_2x_1x_4-\pi_{42}u'_1u_3x_2x_4-\pi_{23}u'_1u_4x_2x_3=0.$$

When $x_1=0$, we have

$$x_2=0, \text{ and } \pi_{23}u_4x_3+\pi_{42}u_3x_4=0;$$

and when $x_2=0$, we have

$$x_1=0, \text{ and } \pi_{13}u_4x_3-\pi_{14}u_3x_4=0.$$

The equations of these three lines

$$d\begin{cases} x_1=0, \\ \pi_{23}u_4x_3+\pi_{43}u_3x_4=0; \end{cases}$$

$$d'\begin{cases} x_2=0, \\ \pi_{13}u_4x_3-\pi_{14}u_3x_4=0; \end{cases}$$

$$e\begin{cases} x_1=0, \\ x_2=0, \end{cases}$$

being independent of the coördinates of the plane u', represent the degenerate k^3 common to all the quadrics which correspond to all the lines in u. e is the common generator, while d and d' are the common directrices.

16. Every plane u, besides determining uniquely a space cubic k^3, contains a conic γ^2 of the complex, the conic to which all the complex lines in the plane are tangent. All these lines, ∞ in number, correspond to the points on k^3; or in other words, all the points on a cubic curve of the complex determined by a plane go into lines tangent to the conic of the complex in that plane.

17. To obtain the equations of this conic γ^2, write the equations of the polar planes of the points on k^3 (Art. 11) with respect to F_1 and F_2, respectively,

$$\sum_{i=1}^{4} a_iu_i \prod_{j=1}^{4} (a_j\lambda-b_j)x_i=0,$$

$$\sum_{i=1}^{4} b_iu_i \prod_{j=1}^{4} (a_j\lambda-b_j)x_i=0 \qquad [i\neq j].$$

These two planes intersect in a line which we shall designate by L. For every value of λ, there is a point on k^3 and there is corresponding to it a line given by L. *All these lines of the system L lie in the plane u*, for the matrix

$$\left\| \begin{matrix} a_1u_1B_2B_3B_4 & a_2u_2B_3B_4B_1 & a_3u_3B_4B_1B_2 & a_4u_4B_1B_2B_3 \\ b_1u_1B_2B_3B_4 & b_2u_2B_3B_4B_1 & b_3u_3B_4B_1B_2 & b_4u_4B_1B_2B_3 \\ u_1 & u_2 & u_3 & u_4 \end{matrix} \right\|$$

[where $B_i=a_i\lambda-b_i$] of the coefficients of the two equations of L and those of the equation of the plane u is of rank 2. To show this is merely algebraic work which any one may easily verify.

18. The line L pierces the plane x_1 and x_2 respectively in

$$R_1\equiv(0 : \pi_{34}u_3u_4B_2 : \pi_{42}u_4u_2B_3 : \pi_{23}u_2u_3B_4),$$
$$R_2\equiv(\pi_{34}u_3u_4B_2 : 0 : \pi_{41}u_4u_1B_3 : \pi_{13}u_1u_3B_4).$$

The plane determined by R_1, R_2 and $A(0:0:0:1)$ is given by

$$p \equiv \pi_{41}u_1B_2B_3x_1+\pi_{42}u_2B_3B_1x_2+\pi_{43}u_3B_1B_2x_3=0$$

or

$$\sum_{i=1}^{3} \pi_{4i}u_i \prod_{j=1}^{3} (a_j\lambda-b_j)x_i=0 \qquad [i\neq j].$$

This equation involves λ in the second degree and is therefore a system of planes enveloping a quadric cone whose vertex is the point $A_4(0:0:0:1)$. Therefore, the equations

$$p=0,\ u=0$$

represent the system of lines tangent to the conic γ^2 of the complex in the plane u and these lines are lines of the complex.

19. This system of rays is none other than the system L given by the equations of Art. 17. Thus, *when a plane is given we can easily write down the equations of the system of lines of the complex lying in it.*

20. The system of complex lines in the plane u $(0, u_2, u_3, u_4)$ is made up of two flat pencils of the first class, one of which corresponding to the points on the conic c_2 (Art. 13) in x_1 has its center at A_1 and the other corresponding to the line d (Art. 14) has its center at the point

$$A'_1\equiv(0:\pi_{12}\pi_{34}u_3u_4:\pi_{13}\pi_{42}u_4u_2:\pi_{14}\pi_{23}u_2u_3)$$

whose transform is the line d itself. To obtain the equations of the first pencil, one needs only to put $u_1=0$ in

$$p=0,\ u=0$$

given in Art. 18 or in the equations given in Art. 17. The equations of the other pencil are those of the polar planes of points of d whose coördinates are given in Art. 14 with respect to F_1 and F_2 and are given by

$$\lambda a_1u_3\pi_{12}\pi_{14}x_1+a_1u_2\pi_{13}\pi_{14}x_2+a_3u_3\pi_{12}\pi_{14}x_3+a_4u_4\pi_{12}\pi_{13}x_4=0,$$
$$\lambda b_1u_3\pi_{12}\pi_{14}x_1+b_2u_2\pi_{13}\pi_{14}x_2+b_3u_3\pi_{12}\pi_{14}x_3+b_4u_4\pi_{12}\pi_{13}x_4=0.$$

For every value of λ there is a ray of the pencil, and every ray of this pencil pierces the plane x_1 in the same point A_1 whose coördinates are given in Art. 20. The polar planes of this point A'_1 with respect to F_1 and F_2 are

$$a_2u_3u_4\pi_{12}\pi_{34}x_2+a_3u_2u_1\pi_{13}\pi_{42}x_3+a_4u_2u_3\pi_{14}\pi_{23}x_4=0,$$
$$b_2u_3u_4\pi_{12}\pi_{34}x_2+b_3u_2u_4\pi_{13}\pi_{42}x_3+b_4u_2u_3\pi_{14}\pi_{23}x_4=0$$

which intersect in a line which is no other than d itself.

21. Similarly, we can show without any difficulty that the system of complex lines in a plane containing an edge, A_1A_2, of the tetrahedron of reference is made up of two flat pencils, one with center at A_1 and the other with center at A_2.

22. Now we examine further the relations between the lines and points in any plane u and the points and bisecants of the space cubic k^3 which u determines. Since every point on k^3 transforms itself into a line of the complex in u and every

point in u transforms itself into a line cutting across k^3 twice, let l_1 and l_2 be two lines in u corresponding to the points L_1 and L_2 on k^3, and then the point P in which l_1 and l_2 intersect goes into the line p cutting across k^3 in L_1 and L_2. Thus, a one-to-one correspondence exists between the points in u considered as the intersection of two tangents to the conic γ_2 of the complex and the bisecants cutting across the cubic k^3 in the two points to which the tangents to γ_2 correspond. If L_1 and L_2 approach coincidence, the rays l_1 and l_2 also approach coincidence. The p becomes a tangent to k^3 while the point P in u moves up to the conic γ^2. We can easily conclude that

A point without γ^2 goes into a real bisecant of k^3; on γ^2, a tangent; within γ^2, a line with no point in common with k^3.

23. We have seen that every conic γ'^2 by our method of transformation, goes into a quartic scroll. In particular, if γ'^2 is coincident with γ^2 every point of which goes into a tangent to k^3, the surface is a developable with k^3 as the cuspidal edge. But in general, γ'^2, when not coincident with γ^2, is cut by every tangent to γ^2 in two points; therefore, every point on k^3, in general, is a double point on the corresponding quartic surface. In other words, k^3 is a double curve. We can easily see that *the ruled sextic surface coming from a plane cubic contains k^3 as triple curve*, and that, *in general, the ruled surface of degree 2n coming from a plane curve of degree n contains k^3 as n-fold curve.*

24. From the figure of the preceding article we see that corresponding to P_1 on the conic γ'^2, which is the intersection of the complex lines l_1 and l_2, is the ruling p_1 on the corresponding quartic, cutting across k^3 in L_1 and L_2. But the ray l_2 intersects γ'^2 again in P_2 which is on another complex line l_3; therefore, on the quartic surface, from the point L_2 on k^3 issues another ruling p_2 meeting k^3 again in the point L_3. Continuing the process, we see that *from every point on k^3 go two generators of the quartic meeting the curve again in two different points, from each of which goes another generator.* But if a ray of the complex is tangent to the given conic γ'^2, i.e., cuts it in two coincident points, then corresponding to the two coincident points P_4 and P_5 common to γ'^2 and l_5 are two coincident generators from the point L_5 on k^3 on the quartic. Then L_5 is a pinch-point on the surface. We may thus conclude that *the existence of a line of the complex tangent to a given plane curve means the existence of a pinch-point on the corresponding surface,* and that, in particular, *a ruled quartic with a double space cubic which does not degenerate can have no more than four pinch-points.* If γ'^2 intersects γ^2 in two real points, two of the pinch-points become imaginary, for the two conics can have only two real common tangents.

SECTION III

25. The most general equation of the ruled quartic from any conic is given in Art. 9. Let the lines p'_{ik}, p''_{ik}, p'''_{ik} be three lines of the complex in the plane u, i.e., form a circumscribed triangle to γ^2, and further let P', P'', P''' be the three points on k^3 to which p'_{ik}, p''_{ik}, p'''_{ik} correspond respectively. Finally, let p'_{ik} and p''_{ik} intersect in R''', p''_{ik} and p'''_{ik} in R', p'''_{ik} and p'_{ik} in R''. Without loss of generality, let a given conic in the plane pass through R'' and R''' but not R'. Then the corresponding quartic surface is given by

$$A_{11}Q'^2+2A_{12}Q'Q''+2A_{13}Q'''Q'+2A_{23}Q''Q'''=0$$

where the Q's are the left members of the equations of the quadric cones corresponding to the lines p'_{ik}, p''_{ik}, p'''_{ik} (Art. 9). This equation can be written

$$\sum_1^4 \pi_{ik}\pi_{jl}\left[\sum_1^3 A_{rs}p^{(r)}{}_{ik}p^{(s)}{}_{jl}\right]x_ix_kx_jx_l=0 \qquad [A_{22}=A_{33}=0].$$

26. Now corresponding to the points R', R'', R''' are three lines r'_{ik}, r''_{ik}, r'''_{ik} forming a triangle with its vertices P', P'', P''' on k^3. Let the plane of this triangle be denoted by u' and its complex cubic by k'^3. The above quartic surface is intersected by u' in a conic γ''^2 through P'' and P''' and two straight lines which are rulings of the surface corresponding to the points R'' and R'''. Note that R', R'', R''', being three points whose transforms are three lines in u', are three points of the space cubic k'^3 which u' determines. Then the conic γ''^2 in u' will go into a quartic surface which is intersected by u in the conic γ'^2 and the lines p''_{ik} and p'''_{ik} which are rulings of the surface. Therefore, we conclude:

27. *Any conic through two vertices of a triangle whose sides are complex lines* (*or through two points on any complex space cubic*) *goes into a quartic surface all of whose generators intersect another conic through two vertices of another triangle whose sides are complex lines* (*or through two points on another complex cubic*). *The quartic corresponding to the latter conic will have its rulings all going through the former conic.* This class of quartics we designate as class I (III of Sturm).

28. But if the given conic γ'^2 goes through R' also, we have class II (IV of Sturm). The equation of this quartic surface is

$$A_{12}Q'Q''+A_{13}Q'Q'''+A_{23}Q''Q'''=0$$

which is the same as the equation of the quartic surface given in Art. 9 after having imposed the condition that $r\neq s$. This surface is intersected by the plane u' in four straight lines three of which are r'_{ik}, r''_{ik}, r'''_{ik} and the fourth which we designate by s_{ik} is the very line whose corresponding quadric is intersected by u in the given conic γ'^2. Every plane through s_{ik} cuts the surface in three generators and this is to be expected, for there are an infinite number of triangles with vertices on γ'^2 and sides tangent to γ^2 [C. Smith, *Conic Sections*, p. 275].

29. These two classes have a space cubic for double curve. Class I has all its generators going through a conic with two points in common with the double curve and class II has all its generators going through a straight line which has no point in common with the double curve. The former reciprocates into one of the same kind, while the latter reciprocates into one with a triple line [see below, class *VIII*].

30. Now let the plane u pass through A_1, a vertex of the fundamental tetrahedron. The k^3 of this plane is made up of the conic c^2 in the plane x_1 and a straight line d, and the system of complex lines is made up of two flat pencils A_1 and A'_1 (Art. 20). Every conic γ'^2 not going through A_1 in u transforms itself into a quartic with c^2 as double conic and d as double line. From every point on d issue two generators meeting c^2 in two different points and vice versa. It can be easily seen that there are in general two pinch-points on c^2 and two on d, for there can be in general two tangents from A_1 and two from A'_1 to γ'^2.

31. Let p'_{ik} and p''_{ik} be two rays of A'_1 and $p'''_{ik}(x'''_2 : x'''_3 : x'''_4 : 0 : 0 : 0)$ be a ray of A_1, meeting p'_{ik} in R'' and p''_{ik} in R'. Without loss of generality we can let a given conic γ^2 pass through R' and R'' (but not through A_1 nor A'_1) and the equation of its corresponding quartic is the same as that of class I given at the end of Art. 25 after making the necessary changes in the p_{ik}'s. This quartic is intersected by the plane u' determined by the line d (corresponding to A'_1) and the lines r'_{ik} and r''_{ik} (corresponding to R' and R'') in a conic γ'^2 and two generators (r'_{ik} and r''_{ik}). This conic γ'^2 goes back into a quartic whose section by u is no other than the given conic γ'^2 and the two lines p'_{ik} and p''_{ik}. Therefore, *this quartic has all its generators going through a conic.* This is class III (V of Sturm). If γ'^2 is tangent to the line $A_1A'_1$, one of the pinch-points kn c^2 and one of the pinch-points on d fall together at the point common to c^2 and d.

32. If γ'^2 goes through A'_1 also, its transform is of class IV (VI of Sturm) whose equation can be easily obtained by putting $A_{33}=0$ in the equation of class III. The plane section by u' is made up of four straight lines r'_{ik}, r''_{ik}, the line d, and a fourth line ρ_{ik} whose corresponding quadric is cut by u in the given conic γ'^2 itself. This quartic, every one of whose generators, besides meeting c^2 and d, meets this line ρ_{ik}, reciprocates into one with a triple line [see class IX below]. Note that there is one pinch-point on d and two on c^2, for there is only one ray of the pencil A'_1 and two of the pencil A_1 tangent to γ'^2.

33. If the edge A_1A_2 of the fundamental tetrahedron lies in the plane u, any conic γ'^2 in it not going through A_1 and A_2, the centers of the two flat pencils making up the complex conic γ^2 (Art. 21), transforms itself into a quartic surface whose double cubic is $d+d'+e$. Letting p'_{ik} and p''_{ik} be two rays of pencil A_2 and p'''_{ik} be a ray of pencil A_1, we have

$$p'_{13}=p'_{14}=p'_{34}=0,\ p''_{13}=p''_{14}=p''_{34}=0,\ p'''_{23}=p'''_{42}=p'''_{34}=0;$$

and putting these in the equation of the quartic in Art. 9, we have the resulting equation representing the quartic in question. This is class V (VII of Sturm). There are in general two pinch-points on each of the lines d and d'. e, to which every point on A_1A_2 but not A_1 and A_2 is transformed, for γ'^2 has two points, real or imaginary, in common with A_1A_2. But if the given conic γ'^2 is tangent to A_1A_2 e is a cuspidal ruling and d and d' have each only one pinch-point.

34. To obtain the next class, VI (I of Sturm), of quartics which have only two double lines and no double generator, we take a cubic curve without a double point in the plane u through the points A_1 and A_2. The corresponding surface is a sextic made up of the two planes x_1 and x_2 corresponding to A_1 and A_2 respectively and a quartic.

35. Let the equation of a cubic cone with its vertex at $A_4(0 : 0 : 0 : 1)$ and with two elements one through A_1 and the other through A_2 be

$$\sum_1^3 A_{rst}x_rx_sx_t=0 \qquad\qquad [A_{111}=A_{222}=0].$$

The plane section of this cone by the plane $u(0 : 0 : u_3 : u_4)$ is a cubic curve through A_1 and A_2. Letting p'_{ik}, p''_{ik}, p'''_{ik} be the coördinates of the lines in which the planes x_1, x_2, x_3 intersect u, we have

$$\begin{aligned} &p'_{ik}(0 : 0 : 0 : 0 : p'_{42} : p'_{23}) && [\text{where } p'_{42}=u_3;\ p'_{23}=u_4],\\ &p''_{ik}(0 : p''_{13} : p''_{14} : 0 : 0 : 0) && [\text{where } p''_{13}=u_4;\ p''_{14}=-u_3],\\ &p'''_{ik}(p'''_{12} : 0 : 0 : 0 : 0 : 0 :) && [\text{where } p'''=u_4]. \end{aligned}$$

Then the quadrics corresponding to these lines are respectively

$$\begin{aligned} Q' &\equiv x_2[\pi_{42}p'_{43}x_4+\pi_{23}p'_{23}x_3]=0,\\ Q'' &\equiv x_1[\pi_{13}p''_{13}x_3+\pi_{14}p''_{14}x_4]=0,\\ Q''' &\equiv \pi_{12}p'''_{12}x_1x_2=0. \end{aligned}$$

Therefore, the sextic surface corresponding to the cubic curve is given by

$$\sum_1^3 A_{rst}Q^{(r)}Q^{(s)}Q^{(t)}=0 \qquad\qquad [A_{111}=A_{222}=0].$$

which, when expanded, is

$$x_1x_2\,[\text{a quartic factor}]=0.$$

36. The factor enclosed in the bracket equated to zero gives a quartic surface. If the cubic has a double point, this surface is of class V, for it has d and d' for double directrices and has a double generator which, however, is different from e. But if we impose upon the A_{rst}'s the condition that the cubic be without a double point, the surface will be without a double generator.

37. This surface has four pinch-points on each of the two double directrices, for, the class of the cubic curve without a double point being six, there can be drawn from a point on it four tangents exclusive of the one at the point itself.

38. Note that the equation of the degenerate sextic surface given in Art. 35 could have been obtained from the sextic equation in Art. 9 by putting A_{111}, A_{222}, p'_{12}, p'_{13}, p'_{14}, p'_{34}, p''_{12}, p''_{34}, p''_{42}, p''_{23}, p'''_{13}, p'''_{14}, p'''_{34}, p'''_{42}, p'''_{23} all equal to zero [Art. 35].

39. Now if we put in the equation of the cone in Art. 35

$$A_{111}=A_{112}=A_{113}=0,$$

the result is that of a cone with

$$x_2=0,\ x_3=0$$

as double element, thus giving a plane curve in u with a double point at A_1. With A_{111}, A_{112}, A_{113}, p'_{12}, p'_{13}, p'_{14}, p'_{34}, p''_{12}, p''_{34}, p''_{42}, p''_{23}, p'''_{13}, p'''_{14}, p'''_{34}, p'''_{42}, p'''_{23} all put equal to zero, the left member of the sextic equation in Art. 9 factors into x^2_1 [a quartic factor]. $x^2_1=0$ represents a double plane corresponding to the double point A_1. The quartic factor equated to zero represents a quartic surface which has the line d' as a triple line and d as a simple line. From every point on d' issue three rulings corresponding to the three points in which every line of the pencil A_2 intersects the given cubic; and from every point on d issues only one generator corresponding to the point in which every ray of the pencil A_1 meets the cubic besides A_1 itself. This class of quartics is class VII (X of Sturm). Since from the point A_2 which is not on the curve only four tangents can be drawn to the curve, the triple line has four pinch-points, i.e., points at which two of the three generators become coincident.

40. To get class VIII (IX of Sturm), we must return to the case where the plane u passes through the vertex A_1 of the fundamental tetrahedron. Remembering that the coördinates of A'_1 are [Art. 20].

$$(0 : \pi_{12}\pi_{34}u_3u_4 : \pi_{13}\pi_{42}u_4u_2 : \pi_{14}\pi_{23}u_2u_3)$$

and letting $A'_1A''_1$ and $A_1A''_1$ be the lines of intersection of u with x_1 and x_2 respectively, thus giving $(0 : 0 : u_4 : -u_3)$ for the coördinates of A''_1, we have

$$\begin{aligned}
A'_1A''_1 &\equiv p'_{ik}(0 : 0 : 0 : u_2 : u_3 : u_4),\\
A_1A''_1 &\equiv p''_{ik}(0 : u_4 : -u_3 : 0 : 0 : 0),\\
A_1A'_1 &\equiv p'''_{ik}(\pi_{12}\pi_{34}u_3u_4 : \pi_{13}\pi_{42}u_4u_2 : \pi_{14}\pi_{23}u_2u_3 : 0 : 0 : 0).
\end{aligned}$$

To these three lines correspond the following quadrics, respectively:

$$\begin{aligned}
Q' &\equiv \pi_{34}u_2x_3x_4+\pi_{42}u_3x_4x_2+\pi_{23}u_4x_2x_3=0,\\
Q'' &\equiv x_1(\pi_{13}u_4u_3-\pi_{14}u_3x_4)=0,\\
Q''' &\equiv x_1(\pi^2_{12}\pi_{34}u_3u_4x_2+\pi^2_{13}\pi_{42}u_2u_4x_3+\pi^2_{14}\pi_{23}u_2u_3x_4)=0.
\end{aligned}$$

Then the equation of the sextic surface corresponding to any cubic in u with a double point at A_1 is given by

$$\sum_1^3 A_{rst}Q^{(r)}Q^{(s)}Q^{(t)}=0. \qquad [A_{111}=A_{112}=A_{113}=0].$$

The result of expanding this equation which is the same as that obtained from the sextic equation in Art. 9 by putting $A_{111}=A_{112}=A_{113}=0$ and by assigning to the p_{ik}'s their respective values given above, is an equation whose left member is made up of two factors, x^2_1 and a quartic factor. $x^2_1=0$ gives a double plane corresponding to the double point A_1, and the quartic factor equated to zero represents a quartic surface which has a triple line coincident with d from every point of which issue three generators meeting the conic c^2 in x_1 in three different points. Quartics of this class have in general four pinch-points on d, and their reciprocals are of class II already considered.

41. If the cubic goes through A'_1, then $A_{222}=0$, and the corresponding quartic surface has one of its generators coincident with the triple line d. This is of class IX (XI of Sturm).

SECTION IV

42. So far we have been able to obtain nine classes of ruled quartics; to obtain the remaining classes we need to alter the machinery of transformation. Let the fundamental quadrics touch each other along a straight line, and, for the sake of simplicity, let their equations be

$$S_1 \equiv a_{11}x^2_1+a_{22}x^2_2+2a_{23}x_2x_3+2a_{14}x_1x_4=0,$$
$$S_2 \equiv b_{11}x^2_1+b_{22}x^2_2+2b_{23}x_2x_3+2b_{14}x_1x_4=0 \qquad [a_{23} : b_{23}=a_{14} : b_{14}],$$

the line of tangency being $x_1=0$, $x_2=0$. Any point Y (y_1, y_2, y_3, y_4) in space goes into the line

$$(a_{11}y_1+a_{14}y_4)x_1+(a_{22}y_2+a_{23}y_3)x_2+a_{23}y_3x_3+a_{14}y_1x_4=0,$$
$$(b_{11}y_1+b_{14}y_4)x_1+(b_{22}y_2+b_{23}y_3)x_2+b_{23}y_2x_3+b_{14}y_1x_4=0,$$

intersecting the line $x_1=0$, $x_2=0$. Therefore, all the lines thus obtained cut across the line of tangency of the two given quadrics. To the point $(0 : 0 : l : m)$ on this line of tangency correspond the common tangent planes to S_1 and S_2 at the point. Every point in x_1 goes into a ray through $A_4(0 : 0 : 0 : 1)$ and every point in x_2 goes into a ray through A_3 $(0 : 0 : 1 : 0)$. Since there is nothing special about the plane x_4 except that it is a face of the tetrahedron of reference, we shall find the remaining classes of quartics corresponding to curves in this plane.

43. The system of common tangent planes along $x_1=0$, $x_2=0$ is intersected by the plane x_4 in a flat pencil of rays with center at A_3. Since any point in x_4 is on one of these lines and hence in one of these common tangent planes, its corresponding ray must cut across the line x_1x_2 at the point at which the plane containing the given point is tangent to the two quadrics; but every point on $x_1=0$, $x_4=0$ goes into the same line $x_2=0$, $x_3=0$. Therefore, every conic γ'_2 in x_4 goes into a quartic surface with a double ruling $x_2=0$, $x_3=0$ which corresponds to the two points which γ'^2 has in common with $x_1=0$, $x_4=0$. From every point of the line of tangency of S_1 and S_2 issue two generators of the quartic surface, corresponding to the two

points in which every ray of the pencil A_3 cuts γ'^2; therefore, this line is a double line. Furthermore, it is a line along which two double lines coincide, for every ray of the pencil A_3 is a line in which two coincident planes tangent to S_1 and S_2 intersect x_4. This quartic is of class X (VIII of Sturm), and its equation is obtained as follows:

Let

$$\sum_1^3 A_{ij}x_ix_j=0$$

be the equation of any cone whose plane section in x_4 is a conic γ'^2. Any line

$$u_1x_1+u_2x_2+u_3x_3=0,\ x_4=0$$

goes into a quadric

$$(u_3\varphi_{22,\ 11}+u_2\varphi_{11,\ 23})x_1x_2+u_3\varphi_{23,\ 11}x_1x_3+u_2\varphi_{23,\ 22}x_2+u_3\varphi_{22,\ 14}x_2x_4=0$$

where

$$\varphi_{ij,\ kl}=a_{ij}b_{kl}-a_{kl}b_{ij}$$

and

$$\varphi_{ij,\ kl}=-\varphi_{kl,\ ij}.$$

Putting $u_1=1$, $u_2=u_3=0$, we have

$$Q_1\equiv\varphi_{22,\ 23}x^2_2=0,$$

the quadric corresponding to the line $x_1=0$, $x_4=0$. Putting $u_2=1$, $u_1=u_3=0$, we have

$$Q_2\equiv\varphi_{11,\ 23}x_1x_2=0,$$

the quadric corresponding to $x_2=0$, $x_4=0$. Finally, putting $u_3=1$, $u_1=u_2=0$, we have the quadric

$$Q_3\equiv\varphi_{11,\ 23}x_1x_2+\varphi_{11,\ 23}x_1x_3+\varphi_{14,\ 22}x_2x_4=0$$

corresponding to the line $x_3=0$, $x_4=0$. Then the equation of the quartic is given by

$$\sum_1^3 A_{ij}Q_iQ_j=0$$

or, when expanded,

$$A_{11}\varphi^2_{22,\ 23}x^4_2+2A_{13}\varphi_{11,\ 22}\varphi_{22,\ 23}x_1x^3_2+2A_{12}\varphi_{11,\ 23}\varphi_{22,\ 23}x_1x^3_3+2A_{13}\varphi_{22,\ 23}\varphi_{14,\ 22}x^3_2x_4$$
$$+(A_{22}\varphi^2_{11,\ 23}+2A_{23}\varphi_{11,\ 22}\varphi_{11,\ 23}+A_{33}\varphi^2_{11,\ 22})x^2_1x^2_2+A_{23}\varphi^2_{11,\ 23}x^2_1x^2_3+A_{33}\varphi^2_{14,\ 22}x^2_2x^2_4+2\varphi_{11,\ 23}(A_{33}\varphi_{11,\ 22}+A_{23}\varphi_{11,\ 23})x^2_1x_2x_3+2\varphi_{42}(A_{33}\varphi_{11,\ 22}+A_{23}\varphi_{11,\ 23})$$
$$x_1x^2_2x_4+2A_{33}\varphi_{11,\ 23}\varphi_{14,\ 22}x_1x_2x_3x_4=0.$$

44. Putting $x_2=0$, we have $x^2_3=0$, and vice versa, showing that the line $x_2=0$, $x_3=0$ is a double generator. Putting $x_4=0$, we have the curve of intersection of the surface in x_4, having a node at A_1 and a tacnode at A_3 with the line A_2A_3 as tangent.

45. The quartics of class XI (II of Sturm) which have two coincident double directrices but no double generator, can be obtained from a cubic in x_4, tangent to the line A_2A_3 [$x_1=0$, $x_4=0$] at A_3. This curve must not have a double point, for then the corresponding surface would have a double generator; nor must it go

through the point $A_1(1 : o : o : o)$ whose transform is the plane x_1, for then it would be impossible to obtain a quartic surface. Then to the given cubic we have corresponding a sextic surface which is made up of the plane x_2 counted twice corresponding to the point A_3 of contact between the curve and A_2A_3, and a quartic which has no double ruling.

46. Let

$$\sum_1^3 A_{ijk}x_ix_jx_k=0,\ x_4=0 \qquad [A_{333}=A_{233}=0]$$

be such a cubic curve, which transforms itself into the sextic surface

$$\sum_1^3 A_{ijk}Q_iQ_jQ_k=0 \qquad [A_{333}=A_{233}=0]$$

where the Q's have the same meaning as those of Art. 43. This equation, if expanded, would be

$$x^2_2[\text{a quartic factor}]=0.$$

The bracketed factor equated to zero is the equation of the required quartic surface.

47. If, in addition, A_{133} is put equal to zero, we have a cubic curve with a node at A_3, and the corresponding surface is of class XII (XII of Sturm, which has the line $x_1=0$, $x_2=0$ for triple line, that is, a line through which three sheets of the surface pass. If the double point at A_3 becomes a cusp, only one of the sheets passes through the triple line, while the other two unite into a cuspidal sheet. This is a special case of the above but Salmon made a separate class (V according to his enumeration) of it.

48. This concludes the study and classification of ruled quartic surfaces. It may be added that classes could have been obtained from plane quartics with their singular points at one or two of the vertices of the fundamental tetrahedron, from space cubis not belonging to the complex through one vertex of the fundamental tetrahedron and also from space quartics through all the four vertices of the fundamental tetrahedron, but none can come from quintic curves or curves of higher order, plane or space.

Many thanks are due to Professor D. N. Lehmer, without whose kind and patient guidance this work would have been an impossibility.

Vol. 2. 1909–1921.

1. Overstreet, The Dialectic of Plotinus .... .25
2. Hocking, Two Extensions of the Use of Graphs in Formal Logic. 20 figures in text .... .15
3. Hocking, On the Law of History .... .20
4. Adams, The Mystical Element in Hegel's Early Theological Writings .35
5. Parker, The Metaphysics of Historical Knowledge .... .85
6. Boas, An Analysis of Certain Theories of Truth 1.00

Vol. 3. 1918–

1. Rieber, Footnotes to Formal Logic. Pp. 1–177. Cloth, $2.00; paper, $1.50. Carriage extra. Weight, cloth, 2 lbs.; paper, 1½ lbs.
2. Prall, A Study in the Theory of Value. Pp. 197–290 .... 1.00

INDEPENDENT VOLUMES*

Bacon, Leonard, and Rose, Robert S.	The Lay of the Cid. xiv + 130 pages. Cloth, $1.35. Carriage extra. Weight, 1½ lbs.
Daniel, J. Frank	The Elasmobranch Fishes. xi + 334 pages, 260 illustrations. Cloth, $5.50. Carriage extra. Weight, 2 lbs., 10 oz.
Glover, T. R.	Herodotus. (*In press.*)
Grinnell, Joseph Bryant, Harold C. Storer, Tracy I.	The Game Birds of California. x + 642 pages, 16 colored plates, 94 line drawings. Cloth, $6.00. Carriage extra. Weight, 4½ lbs.
Hart, Walter M.	Kipling the Story Writer. 225 pages. Cloth, $2.25. Carriage extra. Weight, 1½ lbs.
Lewis, C. I.	Survey of Symbolic Logic. vi + 407 pages. Cloth, $3.00. Carriage extra. Weight, 2¾ lbs.
McCormac, Eugene I.	James K. Polk: A Political Biography. x + 746 pages, 2 plates. Cloth, $6.00 Carriage extra. Weight, 3 lbs., 10 oz.
McEwen, George F.	Ocean Temperatures: Their Relation to Solar Radiation and Oceanic Circulation. 130 pages. Paper, $1.50.
Morley, S. Griswold	Anthero de Quental: Sonnets and Poems. xxiv + 133 pages. Quarter-vellum, $2.25.
Moses, Bernard	Spain's Declining Power in South America. xx + 440 pages. Cloth, $4.00. Carriage extra. Weight, 2 lbs., 4 oz.
Petersson, Torsten	Cicero: A Biography. 699 pages. Cloth, $5.00. Carriage extra. Weight, 4 lbs.
Scott, John A.	The Unity of Homer. 275 pages. Cloth, $3.25. Carriage extra. Weight, 1 lb.
Slate, Frederick	The Fundamental Equations of Dynamics. xii + 225 pages. Cloth, $2.00. Carriage extra. Weight, 1½ lbs.
Tolman, Richard C.	Theory of the Relativity of Motion. ix + 242 pages. Cloth, $1.75. Carriage extra. Weight, 1¾ lbs.

* The works listed as independent volumes are also on sale at The Baker and Taylor Co., 354 Fourth Avenue, New York City.